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A diagrammatic expansion for the density correlation function

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Abstract. The density field formalism of Sherrington is used to obtain diagrammatic expansions and thus *complete* expressions for the quantities that enter into the evaluation of the density correlation function. It is shown how these diagrammatic expansions satisfy two important sum rules to all orders. The plasmon mode is obtained in a simple fashion from these expansions.

1. Introduction

In the new method of expansion in quantum many-body theory developed by Edwards *et al.* the task of obtaining the one-body Green function becomes that of solving a small number of coupled diagrammatic equations (Edwards and Sherrington 1967, Bhagavan and Edwards 1967, to be referred to as I and II respectively). Sherrington (1967, to be referred to as III) has shown how the analysis of I can be extended to obtain an expansion for the density correlation function to a given order. In the present work we are concerned with obtaining diagrammatic equations for the case of the density correlation function, thus arriving at *complete* expressions for the quantities that enter into the equations. We will then show that the plasmon mode is readily obtained from these diagrammatic equations, and that further they make it possible to demonstrate in a very simple manner that the expansion satisfies sum rules to all orders.

In the appendix we present a very brief account of the analysis contained in I and II. This may help the reader to follow the arguments given below without immediate reference to I and II.

We restrict ourselves to a translationally invariant 'normal' system of fermions of the same kind at absolute zero temperature. In the notation of II, the Feynman functional measure for the given system becomes

$$P = \exp(\mathrm{i}A) = \exp\left\{\mathrm{i}\sum_{\boldsymbol{k}\alpha E_1}(E_1 - k^2 + \mu)\psi^{\dagger}_{\boldsymbol{k}\alpha E_1}\psi_{\boldsymbol{k}\alpha E_1}\right.$$

$$\left. - \mathrm{i}\sum_{\substack{\boldsymbol{jkl}\alpha\beta \\ E_1E_2E_3}} \tfrac{1}{2}g_k\psi^{\dagger}_{\boldsymbol{j}\alpha E_1}\psi^{\dagger}_{\boldsymbol{l}\beta E_2}\psi_{\boldsymbol{l}+\boldsymbol{k},\beta,E_2+E_3}\psi_{\boldsymbol{j}-\boldsymbol{k},\alpha,E_1-E_3}\right\} \tag{1.1}$$

where g_k, the Fourier transform of the two-body interaction $g(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$, depends only on the absolute value of the linear momentum.

In III Sherrington, by the suitable introduction of an auxiliary real Bose field ϕ, replaces the measure P by its equivalent

$$P^{\text{density}}([\psi],[\psi^\dagger],[\phi]) = \exp(\mathrm{i}A^{\text{density}})$$

$$= \exp\left\{\mathrm{i}\sum_{k\alpha E_1}(E_1 - k^2+\mu)\psi^\dagger_{k\alpha E_1}\psi_{k\alpha E_1} - \mathrm{i}\sum_{kE_1}\tfrac{1}{2}\phi_{kE_1}g_k^{-1}\phi_{kE_1}\right.$$

$$\left. + \sum_{\substack{kl\alpha \\ E_1E_2}}{}' \psi^\dagger_{k\alpha E_1}\psi_{l\alpha E_2}\phi_{k-l,E_1-E_2}\right\} \tag{1.2}$$

and shows that

$$\langle\!\langle \phi_{kE}\phi_{-k,-E}\rangle\!\rangle = -\mathrm{i}g_k - g_k{}^2 \sum_{\substack{ij\alpha\beta \\ E_1E_2}} \langle\!\langle \psi^\dagger_{l\alpha E_1}\psi_{l+k,\alpha,E_1+E}\psi^\dagger_{j\beta E_2}\psi_{j-k,\beta,E_2-E}\rangle\!\rangle \tag{1.3}$$

where the double brackets on the left- and right-hand sides stand for averages over P^{density} and P respectively. The object $g\langle\!\langle\psi^\dagger\psi\psi^\dagger\psi\rangle\!\rangle$ is the bound part of the density correlation function. Therefore the spectrum of the collective excitations of the system is given by the pole of $\langle\!\langle\phi\phi\rangle\!\rangle$ in the energy plane.

The prime on the third summation sign in (1.2) means that $\phi_{0,0}$ is excluded, which of course implies that $\psi^\dagger$ and ψ cannot take on identical sets of indices. This restriction ensures that the crucial condition $\langle\!\langle\phi\rangle\!\rangle = 0$ is satisfied.

Henceforth we shall use the notation

$$\langle\!\langle \phi_{kE}\phi_{-k,-E}\rangle\!\rangle = G^{\text{density}}_{2;kE}.$$

Sherrington proceeds to evaluate G and $G_2{}^{\text{den}}$ to a given order by solving the following equation to that order:

$$\sum_{q\gamma E}\left\{\frac{\partial}{\partial\psi_{q\gamma E}}\left(\mathrm{i}\frac{\partial}{\partial\psi^\dagger_{q\gamma E}}+\frac{\partial A^{\text{den}}}{\partial\psi^\dagger_{q\gamma E}}\right)+\frac{\partial}{\partial\psi^\dagger_{q\gamma E}}\left(\mathrm{i}\frac{\partial}{\partial\psi_{q\gamma E}}+\frac{\partial A^{\text{den}}}{\partial\psi_{q\gamma E}}\right)\right.$$

$$\left.+\eta_{qE}\frac{\partial}{\partial\phi_{-q,-E}}\left(\mathrm{i}\frac{\partial}{\partial\phi_{qE}}+\frac{\partial A^{\text{den}}}{\partial\phi_{qE}}\right)\right\}P^{\text{den}} = 0. \tag{1.4}$$

By introducing the quantities

$$d^{\text{den}}_{1;q\gamma E} = 1 + S^{\text{den}}_{1;q\gamma E}$$

$$d^{\text{den}}_{2;q\gamma E} = 1 + S^{\text{den}}_{2;q\gamma E} \qquad D_{qE} = \eta_{qE} + \mathscr{S}_{qE} \tag{1.5}$$

$$w^{\text{den}}_{1;q\gamma E} = E - q^2 + \mu + R^{\text{den}}_{1;q\gamma}$$

$$w^{\text{den}}_{2;q\gamma E} = E - q^2 + \mu + R^{\text{den}}_{2;q\gamma} \qquad W_{qE} = -E + \eta_{qE}g_q{}^{-1} + \mathscr{R}_q \tag{1.6}$$

and putting

$$G_{q\gamma E} = \frac{d^{\text{den}}_{1;q\gamma E}}{w^{\text{den}}_{1;q\gamma E}} = \frac{d^{\text{den}}_{2;q\gamma E}}{w^{\text{den}}_{2;q\gamma E}} \tag{1.7}$$

$$G^{\text{den}}_{2;qE} = \frac{D_{qE}}{W_{qE}} \tag{1.8}$$

he obtains P^{den} as an expansion in terms of the Gaussian

$$P_0{}^{\text{den}} = \exp\left\{-\mathrm{i}\sum_{k\alpha E_1}\psi_{k\alpha E_1}G^{-1}_{k\alpha E_1}\psi^\dagger_{k\alpha E_1} - \sum_{kE_1}\tfrac{1}{2}\phi_{kE_1}(G^{\text{den}}_{2;kE_1})^{-1}\phi_{-k,-E_1}\right\} \tag{1.9}$$

and polynomials of ψ, $\psi^\dagger$ and ϕ.

The quantity η_{qE} introduced into the above equations has, by definition, the dimension of the square of the interaction g, and is an odd function of its arguments, i.e. $\eta_{qE} = -\eta_{-q,-E}$. The latter property ensures the energy independence of the energy denominators in the subsequent expansions, as will be evident later on. One can, for example, assume that η_{qE} has the form $g_q{}^2\{\theta(E)-\theta(-E)\}$, where θ is the Heaviside step function.

The definitions of the quantities D, W, $\mathscr{R}$ and $\mathscr{S}$ introduced above differ from the corresponding ones in III by the factor η. η corresponds to the ξ of III.

On substituting from (1.5) and (1.6), (1.4) becomes

$$(L^{\text{den}}+K^{\text{den}}+M^{\text{den}})P^{\text{den}} = 0 \tag{1.10}$$

where

$$\begin{aligned} L^{\text{den}} = \sum_{q\gamma E} & \mathrm{i}d^{\text{den}}_{1;q\gamma E}\frac{\partial}{\partial\psi_{q\gamma E}}\frac{\partial}{\partial\psi^{\dagger}_{q\gamma E}} + \mathrm{i}d^{\text{den}}_{2;q\gamma E}\frac{\partial}{\partial\psi^{\dagger}_{q\gamma E}}\frac{\partial}{\partial\psi_{q\gamma E}} \\ & + \mathrm{i}D_{qE}\frac{\partial}{\partial\phi_{-q,-E}}\frac{\partial}{\partial\phi_{qE}} + w^{\text{den}}_{1;q\gamma E}\frac{\partial}{\partial\psi_{q\gamma E}}\psi_{q\gamma E} \\ & - w^{\text{den}}_{2;q\gamma E}\frac{\partial}{\partial\psi^{\dagger}_{q\gamma E}}\psi^{\dagger}_{q\gamma E} - W_{qE}\frac{\partial}{\partial\phi_{-q,-E}}\phi_{-q,-E} \end{aligned} \tag{1.11}$$

$$\begin{aligned} K^{\text{den}} = \sum_{q\gamma E} & -\mathrm{i}S^{\text{den}}_{1;q\gamma E}\frac{\partial}{\partial\psi_{q\gamma E}}\frac{\partial}{\partial\psi^{\dagger}_{q\gamma E}} - \mathrm{i}S^{\text{den}}_{2;q\gamma E}\frac{\partial}{\partial\psi^{\dagger}_{q\gamma E}}\frac{\partial}{\partial\psi_{q\gamma E}} \\ & - \mathrm{i}\mathscr{S}_{qE}\frac{\partial}{\partial\phi_{-q,-E}}\frac{\partial}{\partial\phi_{qE}} - R^{\text{den}}_{1;q\gamma}\frac{\partial}{\partial\psi_{q\gamma E}}\psi_{q\gamma E} \\ & + R^{\text{den}}_{2;q\gamma}\frac{\partial}{\partial\psi^{\dagger}_{q\gamma E}}\psi^{\dagger}_{q\gamma E} + (\mathscr{R}_q - E)\frac{\partial}{\partial\phi_{-q,-E}}\phi_{-q,-E} \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} M^{\text{den}} = \sum_{\substack{q\gamma E \\ k\alpha E_1}}{}' & -\mathrm{i}\phi_{q-k,E-E_1}\psi_{k\gamma E_1}\frac{\partial}{\partial\psi_{q\gamma E}} \\ & + \mathrm{i}\phi_{k-q,E_1-E}\psi^{\dagger}_{k\gamma E_1}\frac{\partial}{\partial\psi^{\dagger}_{\gamma E}} + \mathrm{i}\psi^{\dagger}_{k\alpha E_1}\psi_{k-q,\alpha,E_1-E}\frac{\partial}{\partial\phi_{-q,-E}}. \end{aligned} \tag{1.13}$$

The generalized Hermite operator L^{den} annihilates the Gaussian $P_0{}^{\text{den}}$, i.e.

$$L^{\text{den}}P_0{}^{\text{den}} = 0.$$

From (1.5), (1.6), (1.7) and (1.8), we obtain

$$G_{q\gamma E} = \frac{1+S^{\text{den}}_{1;q\gamma E}}{E-q^2+\mu+R^{\text{den}}_{1;q\gamma}} = \frac{1+S^{\text{den}}_{2;q\gamma E}}{E-q^2+\mu+R^{\text{den}}_{2;q\gamma}} \tag{1.14}$$

and

$$G^{\text{den}}_{2;qE} = \frac{\eta_{qE}+\mathscr{S}_{qE}}{-E+\eta_{qE}g_q{}^{-1}+\mathscr{R}_q}. \tag{1.15}$$

The spectrum of the collective excitation is given by

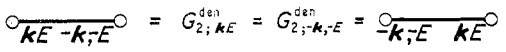
$$E = \eta_{qE}g_q{}^{-1}+\mathscr{R}_q. \tag{1.16}$$

2. Diagrammatic equations

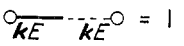
The graphical notation for ψ, $\psi^{\dagger}$, $\partial/\partial\psi$ and $\partial/\partial\psi^{\dagger}$ and the rules for their combination are the same as in II.

We represent ϕ and $\partial/\partial\phi$ respectively by a thick full line and a broken line, both without directional arrows. These we call boson lines. The rules for their combination are the following:

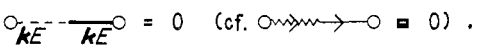
(i) Two thick full lines joined up between any two vertices stand for $G_2{}^{\text{den}}$, e.g.



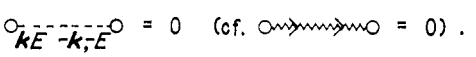
(ii) A full line joining a broken line gives unity, if the former stands to the left of the latter:



(iii) If in (ii) the ordering of the lines is reversed then the result is zero:



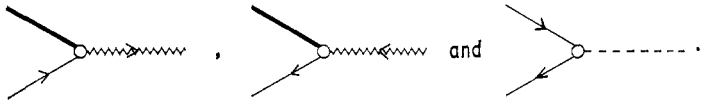
(iv) Two broken lines when joined together give zero:



(v) Lines of the fermion type do *not* join up with those of the boson type.

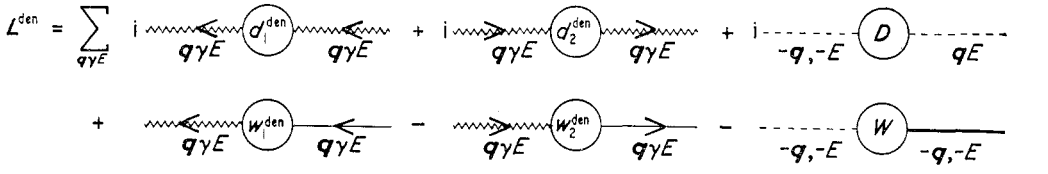
Rules (iii) and (iv) result from the fact that η is an odd function of its arguments, making $\eta_{kE}(\partial/\partial\phi_{-k,-E})\phi_{-k,-E}P_0^{\text{den}}$ and $\eta_{kE}(\partial/\partial\phi_{-k,-E})(\partial/\partial\phi_{kE})P_0^{\text{den}}$ give the same type of results as $(\partial/\partial\psi^{\dagger}_{k\alpha E})\psi^{\dagger}_{k\alpha E}P_0^{\text{den}}$ and $(\partial/\partial\psi^{\dagger}{}_{\alpha E})(\partial/\partial\psi_{k\alpha E})P_0^{\text{den}}$.

The interaction vertex functions for the present case are

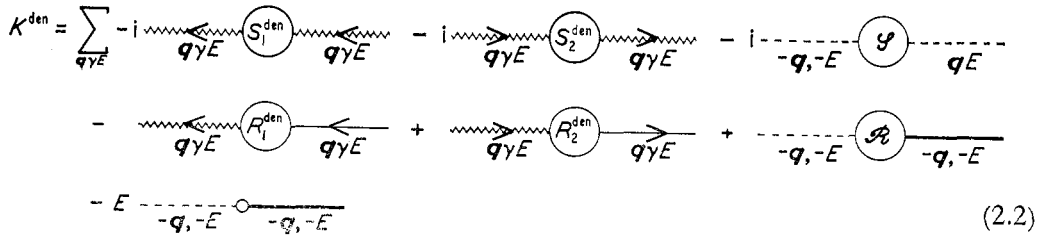


An external wiggly or broken line is always assumed to operate to its right. It is understood that each interaction vertex carries a factor of i, and the vertex with the broken line carries a factor of η.

In terms of the above graphical objects, equations (1.11), (1.12) and (1.13) take on the form

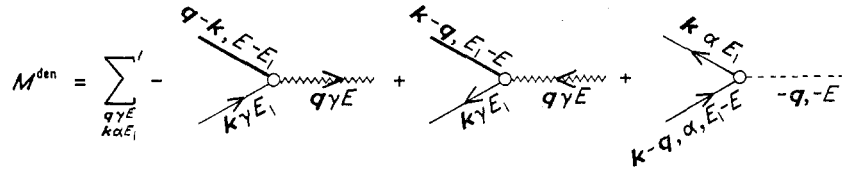


(2.1)



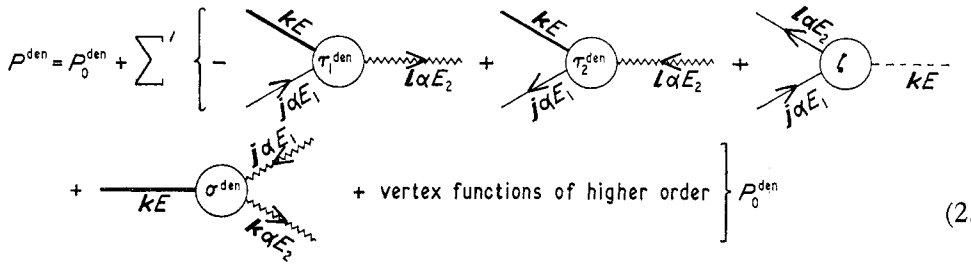
(2.2)

and



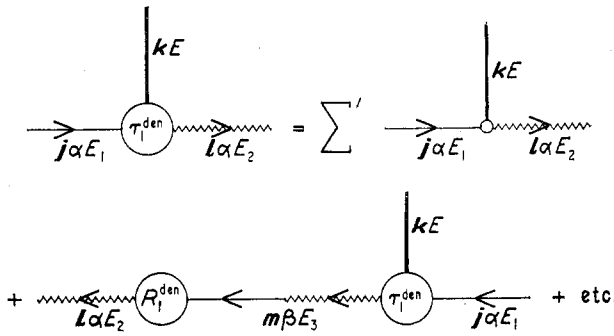
(2.3)

We introduce the expansion

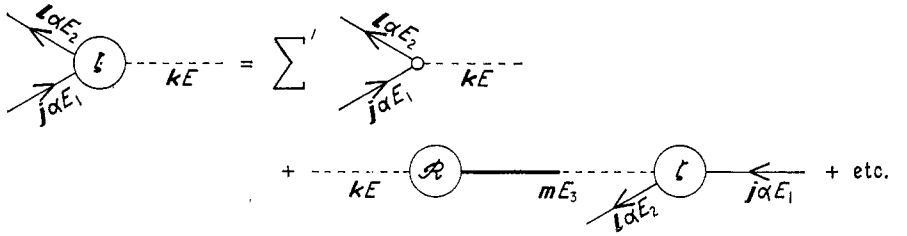


(2.4)

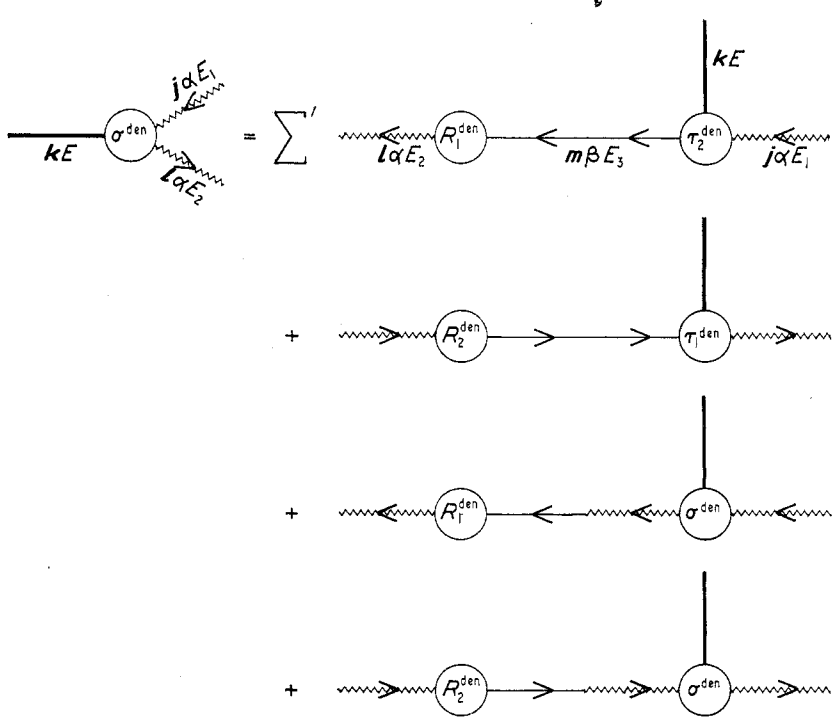
into (1.10), and as in II gather together diagrams with the same set of external lines and set their algebraic sums equal to zero to obtain the following coupled diagrammatic equations:



(2.5)

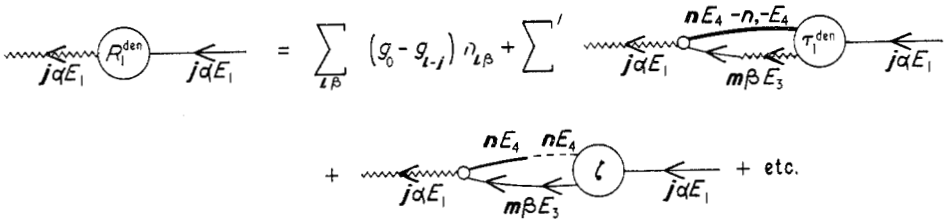


(2.6)



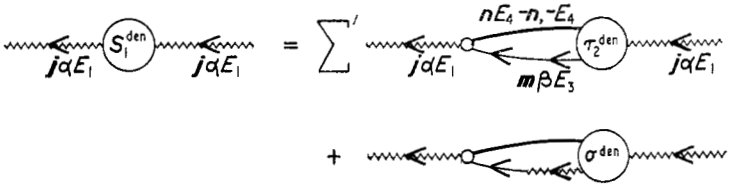
(2.7)

Again, we follow the prescription given in II to identify energy-independent and dependent quantities, thus obtaining the equations for R^{den}, $\mathscr{R}$, S^{den} and $\mathscr{S}$:

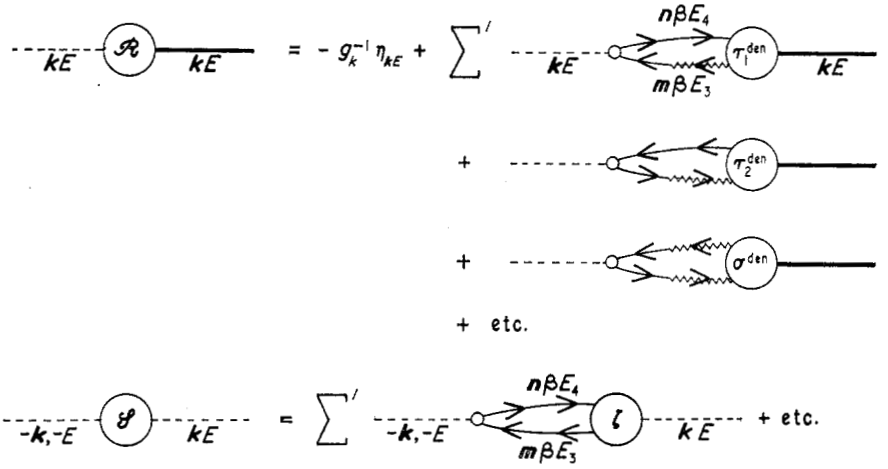


(2.8)

where the first term on the right-hand side is the value of R_1^{den} to the first power in g (cf. 3.28 in II)



(2.9)



(2.10)

(2.11)

The equations for τ_2^{den}, R_2^{den} and S_2^{den} are obtained by reversing the arrows in the equations for τ_1^{den}, R_1^{den} and S_1^{den} respectively.

The indices of the full lines occurring between two vertices are always summed over. At each vertex the energy, momentum and spin are conserved.

The energy denominators are extracted in the same manner as in II and turn out to be independent of energy; the ψ, $\psi^\dagger$ and ϕ lines contribute respectively the terms $(-w_2^{\text{den}})$, w_1^{den} and W, for we have

$$L^{\text{den}}\psi_{k\alpha E}P_0^{\text{den}} = (-w^{\text{den}}_{2;k\alpha E})\psi_{k\alpha E}P_0^{\text{den}}$$

$$L^{\text{den}}\psi^\dagger_{k\alpha E}P_0^{\text{den}} = w^{\text{den}}_{1;k\alpha E}\psi^\dagger_{k\alpha E}P_0^{\text{den}}$$

$$L^{\text{den}}\phi_{kE}P_0^{\text{den}} = W_{kE}\phi_{kE}P_0^{\text{den}}.$$

For example  gives

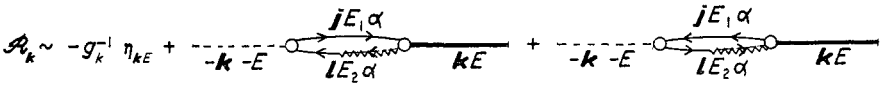
$$(W_{kE}-w^{\text{den}}_{2;j\alpha E_1}+w^{\text{den}}_{1;l\alpha E_2})^{-1}\delta(-E-E_1+E_2)$$
$$=(-E-E_1+E_2+\text{energy-independent terms})^{-1}\delta(-E-E_1+E_2)$$
$$=\text{energy-independent terms}.$$

This circumstance and the form of the equations (2.8) and (2.10) make R_1^{den}, R_2^{den} and $\mathscr{R}$ independent of energy.

A self-consistent iterative procedure is used for solving the above coupled equations for G and G_2^{den}.

3. The plasmon equation

The frequency of the collective mode to order g^2 can be obtained straightaway by inserting the value of $\mathscr{R}$ to that order into the density excitation equation (1.16). From (2.10) we have



$$=g_k^{-1}\eta_{kE}+\sum_\alpha\int \mathrm{d}\boldsymbol{j}\,\mathrm{d}\boldsymbol{l}\,\mathrm{d}E_1\,\mathrm{d}E_2\,\delta(-\boldsymbol{j}+\boldsymbol{l}-\boldsymbol{k})\delta(-E_1+E_2-E)$$
$$\times\eta_{kE}(W_{kE}-w_{2;jE_1\alpha}+w_{1;lE_2\alpha})^{-1}(-\mathrm{i}G_{j\alpha E_1}+\mathrm{i}G_{l\alpha E_2}). \tag{3.1}$$

Putting

$$W_{kE}\sim -E+\eta_{kE}g_k^{-1}$$
$$w_{2;j\alpha E_1}\sim E_1-j^2+\mu$$
$$w_{1;l\alpha E_2}\sim E_2-l^2+\mu$$

(3.1) becomes

$$\mathscr{R}_k\sim -g_k^{-1}\eta_{kE}+\sum_\alpha\int \mathrm{d}\boldsymbol{j}\,\mathrm{d}\boldsymbol{l}\,\delta(-\boldsymbol{j}+\boldsymbol{l}-\boldsymbol{k})\eta_{kE}\frac{n_{j\alpha}-n_{l\alpha}}{\eta_{kE}g_k^{-1}+j^2-l^2}.$$

Hence the plasmon frequency is given by (in units of $h=1$)

$$\nu_k\sim\sum_\alpha\int \mathrm{d}\boldsymbol{j}\,\mathrm{d}\boldsymbol{l}\,\delta(-\boldsymbol{j}+\boldsymbol{l}-\boldsymbol{k})\eta_{kE}(n_{j\alpha}-n_{l\alpha})(\eta_{kE}g_k^{-1}+j^2-l^2)^{-1}.$$

4. Sum rules

The diagrammatic form of the above equations makes it possible to show in a simple manner how the time-derivative sum rule for G and the f sum rule for the density correlation function can be satisfied for any truncation of P^{den} (and thus to any nominal order in R, S and $\mathscr{R}$, $\mathscr{S}$).

(i) In the $\boldsymbol{k}$, t representation, the time-derivative sum rule for G has the form

$$\mathrm{i}\left[\left\{\frac{\partial}{\partial t}G_k(t)\right\}_{t\to 0^+}-\left\{\frac{\partial}{\partial t}G_k(t)\right\}_{t\to 0^-}\right]=-\mathrm{i}\left\{k^2+\sum_{l\beta}(g_0-g_{l-k})n_{l\beta}\right\}.$$

In the $\boldsymbol{k}$, E representation it becomes

$$\int_{-\infty}^{\infty}\mathrm{d}E\,EG_{kE}\{\exp(\mathrm{i}E0^+)-\exp(-\mathrm{i}E0^+)\}=-\mathrm{i}\left\{k^2+\sum_{l\beta}(g_0-g_{l-k})n_{l\beta}\right\}. \tag{4.1}$$

In I, Edwards and Sherrington have shown that since the values of $R^{\text{den}}_{1;k\alpha}$ and $S^{\text{den}}_{1;kE\alpha}$, to the first power in g, are respectively $\Sigma_{l\beta}(g_0-g_{l-k})n_{l\beta}$ and zero, (4.1) takes the form

$$\int_{-\infty}^{\infty}\mathrm{d}E\,S^{\text{den}\,\rangle}_{1;kE\alpha}\{\exp(\mathrm{i}E0^+)-\exp(-\mathrm{i}E0^+)\}=R^{\text{den}\,\rangle}_{1;k\alpha} \tag{4.2}$$

where $S_1^{\text{den}\rangle}$ and $R_1^{\text{den}\rangle}$ are respectively the sum of all nominal orders of S_1^{den} and R_1^{den} excluding the first. Similarly for S_2^{den} and R_2^{den}.

It is clear that (4.2) is always satisfied to any given order, if we substitute to that order from (2.8) and (2.9), because the structure of the equations is such that R^{den} and S^{den} always differ by a factor of G, as for example the following substitution shows:

$$S^{\text{den}}_{1;\boldsymbol{k}E\alpha} = \sum_{\beta} \int \mathrm{d}E_3\, \mathrm{d}E_4\, \mathrm{d}\boldsymbol{n}\, \mathrm{d}\boldsymbol{m}\, G^{\text{den}}_{2;\boldsymbol{n}E_4} G_{\boldsymbol{m}\beta E_3} \tau_1^{\text{den}}(-\boldsymbol{n}, -E_4; \boldsymbol{m}\beta E_3, \boldsymbol{k}\alpha E)$$
$$\times (W_{-\boldsymbol{n},-E_4} - w^{\text{den}}_{2;\boldsymbol{m}\beta E_3} + w^{\text{den}}_{1;\boldsymbol{k}\alpha E})^{-1} \delta_{\alpha\beta}\delta(E_4+E_3-E)\delta(\boldsymbol{n}+\boldsymbol{m}-\boldsymbol{k}).$$

$R^{\text{den}}_{1;\boldsymbol{k}\alpha}$ is the same as $S^{\text{den}}_{1;\boldsymbol{k}E\alpha}$ above, but *without* the factor $G_{\boldsymbol{m}\beta E_3}$.

(ii) Denoting the density correlation function by C, we write

$$C = C_{\text{free}} + C_{\text{bound}}$$

where C_{free} and C_{bound}, which refer respectively to the contributions due to the propagation of two non-interacting free particles and two interacting particles, are given by

$$C_{\text{free}}(\boldsymbol{k}, t) = \sum_{\boldsymbol{l}} \langle\!\langle \psi_{\boldsymbol{l}+\boldsymbol{k}}(t)\psi^{\dagger}_{\boldsymbol{l}+\boldsymbol{k}}(0)\rangle\!\rangle_{\text{free}} \langle\!\langle \psi_{\boldsymbol{l}}(0)\psi_{\boldsymbol{l}}{}^{\dagger}(t)\rangle\!\rangle_{\text{free}}$$
$$= \mathrm{i}^2 \sum_{\boldsymbol{l}} G_{\text{free}}(\boldsymbol{l}+\boldsymbol{k}, t) G_{\text{free}}(\boldsymbol{l}, -t) \tag{4.3}$$

$$C_{\text{bound}}(\boldsymbol{k}, t) = \mathrm{i} g_k \delta(t) + G^{\text{den}}_{2;\boldsymbol{k}}(t). \tag{4.4}$$

The f sum rule when expressed as a zero-time-derivative sum rule has the form

$$\left\{\mathrm{i}\frac{\partial}{\partial t} C_k(t)\right\}_{t=0} = Nk^2 \tag{4.5}$$

where N is the total number of particles in the system.

Now we find that C_{free} by itself exhausts the sum rule (4.5), for from (4.3)

$$\left\{\mathrm{i}\frac{\partial}{\partial t} C_{\text{free}}(\boldsymbol{k}, t)\right\}_{t=0} = \sum_{\boldsymbol{l}} \{(\boldsymbol{l}+\boldsymbol{k})^2 - \boldsymbol{l}^2\} n_l = Nk^2$$

where we have used the relation

$$G_{\text{free};k}(t) = \mathrm{i}\exp(-\mathrm{i}k^2 t)$$

and the fact that n depends upon the absolute value of the momentum. Therefore the zero-time-derivative sum rule reduces to the form

$$\left\{\mathrm{i}\frac{\partial}{\partial t} C_{\text{bound}}(\boldsymbol{k}, t)\right\}_{t=0} = 0. \tag{4.6}$$

Substituting (4.4) into (4.6) and expressing the resulting relation in the $\boldsymbol{k}$, E representation, we obtain

$$\int_{-\infty}^{\infty} \mathrm{d}E\, E G^{\text{den}}_{2;\boldsymbol{k}E} \exp(-\mathrm{i}E0) = 0. \tag{4.7}$$

Writing

$$EG^{\text{den}}_{2;\boldsymbol{k}E} = -\eta_{\boldsymbol{k}E} - \mathscr{S}_{\boldsymbol{k}E}\rangle + (\eta_{\boldsymbol{k}E} g_k^{-1} + \mathscr{R}_{\boldsymbol{k}}) G^{\text{den}}_{2;\boldsymbol{k}E}$$
$$= -\eta_{\boldsymbol{k}E} - \mathscr{S}_{\boldsymbol{k}E}\rangle + \mathscr{R}_{\boldsymbol{k}}\rangle G^{\text{den}}_{2;\boldsymbol{k}E}$$

the sum rule (4.7) becomes

$$\int_{-\infty}^{\infty} \mathrm{d}E\, E \mathscr{S}_{\boldsymbol{k}E}\rangle \exp(-\mathrm{i}E0) = \mathscr{R}_{\boldsymbol{k}}\rangle \tag{4.8}$$

which has the same form as (4.2). From the diagrammatic equations for $\mathscr{S}$, $\mathscr{R}$, τ^{den} and ζ it is seen that (4.8) is satisfied to all orders, since $\mathscr{S}$ and $\mathscr{R}$ always differ by a factor of G.

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Appendix

The Feynman functional integral definition of the one-body Green function is

$$G_{q\gamma E} = \frac{-\mathrm{i}\int \psi_{q\gamma E}\psi^{\dagger}_{q\gamma E} P\Pi\, \delta\psi\, \delta\psi^{\dagger}}{\int P\Pi\, \delta\psi\, \delta\psi^{\dagger}} \tag{A1}$$

where P stands for the Feynman functional measure $\exp(\mathrm{i}A)$. A is Hamilton's principal function, or the action, as it is sometimes called. For the system under consideration

$$A = A_{\text{free}} + A_{\text{int}}$$

$$A_{\text{free}} = \sum_{k\alpha E_1} (E_1 - k^2 + \mu)\psi^{\dagger}_{k\alpha E_1}\psi_{k\alpha E_1}$$

$$A_{\text{int}} = -\sum \tfrac{1}{2} g_k \psi^{\dagger}_{j\alpha E_1}\psi^{\dagger}_{l\beta E_2}\psi_{l+k,\beta,E_2+E_3}\psi_{j-k,\alpha,E_1-E_3}.$$

The usual Feynman perturbation theory can be obtained by expanding P as a series in A_{int}. The difficulties encountered by the usual expansion schemes rest on the fact that G depends, not only in practical evaluation but also formally, on the expansion parameter. Since we wish to avoid these difficulties, we seek to expand P in such a way that G is formally independent of the expansion parameter. Now, it is seen that the solution

$$P = P_0 = \exp\left(-\mathrm{i}\sum_{k\alpha E_1} \psi_{k\alpha E_1} G^{-1}_{k\alpha E_1}\psi^{\dagger}_{k\alpha E_1}\right) \tag{A2}$$

satisfies (A1). If, therefore, we were to expand P in the form

$$\begin{aligned} P &= P_0 + (\text{polynomials in } \psi \text{ and } \psi^{\dagger}) \times P_0 \\ &\equiv P_0 + P_1 + P_2 + \ldots \end{aligned} \tag{A3}$$

and ensure that only P_0 determines G, we will have achieved our object. The choice of (A2) as the zeroth-order approximation to P, and the crucial conditions

$$\int \psi_{q\gamma E}\psi^{\dagger}_{q\gamma E}\left(\sum_{i=1,2,\text{etc.}} P_i\right)\Pi\, \delta\psi\, \delta\psi^{\dagger} = 0 \tag{A4}$$

$$\int \left(\sum_i P_i\right)\Pi\, \delta\psi\, \delta\psi^{\dagger} = 0 \tag{A5}$$

lead to coupled integral equations for G.

P_0 being a Gaussian, the expansion (A3) is in terms of Hermite polynomials in function space. Therefore, we set up a functional differential equation for P, and solve it in terms of Hermite polynomials.

We combine the two infinite sets of equations

$$\mathrm{i}\frac{\partial P}{\partial \psi^{\dagger}_{q\gamma E}} = -\frac{\partial A}{\partial \psi^{\dagger}_{q\gamma E}} P$$

and

$$\mathrm{i}\frac{\partial P}{\partial \psi_{q\gamma E}} = -\frac{\partial A}{\partial \psi_{q\gamma E}} P$$

to obtain

$$\sum_{q\gamma E}\left\{\frac{\partial}{\partial \psi_{q\gamma E}}\left(\mathrm{i}\frac{\partial}{\partial \psi^{\dagger}_{q\gamma E}}+\frac{\partial A}{\partial \psi^{\dagger}_{q\gamma E}}\right)+\frac{\partial}{\partial \psi^{\dagger}_{q\gamma E}}\left(\mathrm{i}\frac{\partial}{\partial \psi_{q\gamma E}}+\frac{\partial A}{\partial \psi_{q\gamma E}}\right)\right\} P = 0. \tag{A6}$$

A generalized Hermite operator of the form

$$L = \sum_{q\gamma E}\left[\frac{\partial}{\partial \psi_{q\gamma E}}\left\{(1+S_{q\gamma E})\mathrm{i}\frac{\partial}{\partial \psi^{\dagger}_{q\gamma E}}+(E-k^2+\mu+R_{q\gamma})\psi_{q\gamma E}\right\}\right.$$

$$\left.+\frac{\partial}{\partial \psi^{\dagger}_{q\gamma E}}\left\{(1+S_{q\gamma E})\mathrm{i}\frac{\partial}{\partial \psi_{q\gamma E}}-(E-k^2+\mu+R_{q\gamma})\psi^{\dagger}_{q\gamma E}\right\}\right] \tag{A7}$$

would be orthogonal to P_0 if we put

$$G_{q\gamma E} = \frac{1+S_{q\gamma E}}{E-k^2+\mu+R_{q\gamma}}. \tag{A8}$$

G as given by (A8) adequately handles all those elementary excitations of the many-body system which arise out of single-particle excitations. (R turns out to be independent of energy.) If we were to write (A6) in such a way as to display explicitly the operator L in it, and then proceed to solve for P in terms of P_0 and higher-order Hermite functions, then we will arrive at equations which permit the evaluation of G in the form (A8). As a result of rewriting (A6) in this fashion, we obtain

$$\left[L-\left\{\sum_{q\gamma E}\frac{\partial}{\partial \psi_{q\gamma E}}\left(S_{q\gamma E}\mathrm{i}\frac{\partial}{\partial \psi^{\dagger}_{q\gamma E}}+R_{q\gamma}\psi_{q\gamma E}\right)+\frac{\partial}{\partial \psi^{\dagger}_{q\gamma E}}\left(S_{q\gamma E}\mathrm{i}\frac{\partial}{\partial \psi_{q\gamma E}}-R_{q\gamma}\psi^{\dagger}_{q\gamma E}\right)\right\}\right.$$

$$-\left\{\sum{}'g_k\left(\psi_{q-k,\gamma,E-E_3}\psi_{l+k,\beta,E_2+E_3}\psi^{\dagger}_{l\beta E_2}\frac{\partial}{\partial \psi_{q\gamma E}}\right.\right.$$

$$\left.\left.\left.+\psi^{\dagger}_{q-k,\gamma,E-E_3}\psi^{\dagger}_{l+k,\beta,E_2+E_3}\psi_{l\beta E_2}\frac{\partial}{\partial \psi^{\dagger}_{q\gamma E}}\right)\right\}\right] P = 0. \tag{A9}$$

The prime on the second summation sign means that ψ and $\psi^{\dagger}$ cannot take on identical sets of indices; the pairs $\psi_1\psi_1{}^{\dagger}$ that have been extracted are thought of as being included in R.

We note that

$$P_0, \qquad \psi_1 P_0, \qquad \psi_1{}^{\dagger}P_0, \qquad \psi_1\psi_2{}^{\dagger}P_0\ (1 \neq 2)$$

$$(\psi_1\psi_1{}^{\dagger}-\mathrm{i}G_1)P_0, \qquad \psi_1\psi_2\psi_3{}^{\dagger}\psi_4{}^{\dagger}P_0\ (1,2 \neq 3,4), \text{ etc.}$$

are eigenfunctions of L and independent Hermite functions of different orders.

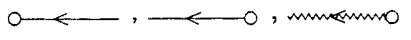
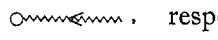
We expand P in the form

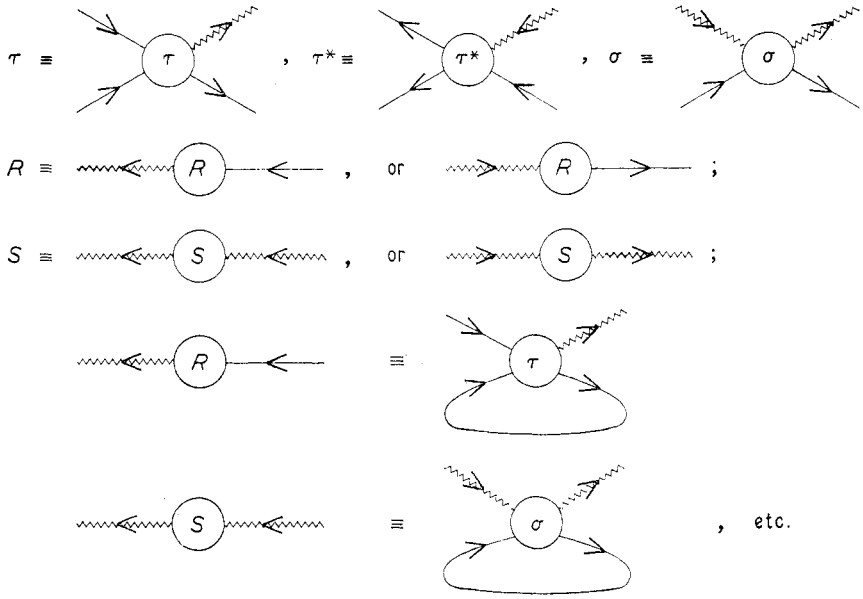
$$P = P_0 + \sum{}'\delta(1+2-3-4)\left\{\tau(1,2,3,4)\psi_1\psi_2\psi_3{}^{\dagger}\frac{\partial}{\partial \psi_4}+\tau^*(1,2,3,4)\psi_1{}^{\dagger}\psi_2{}^{\dagger}\psi_3\frac{\partial}{\partial \psi^{\dagger}_4}\right.$$

$$\left.-\sigma(1,2,3,4)\frac{\partial}{\partial \psi_1{}^{\dagger}}\psi_2\psi_3{}^{\dagger}\frac{\partial}{\partial \psi_4}\right\} P_0 + \text{vertex functions of higher order} \tag{A10}$$

insert it into (A9), and equate the coefficients of each independent Hermite function to zero. We thus get coupled equations for G, R, S, τ and σ, where R and S have the form $\mathrm{Tr}(\tau G)$ and $\mathrm{Tr}(\sigma G)$, respectively.

It is important to note that the prime on the summation sign in (A10), which forbids the ψ from having the same set of indices as the $\psi^\dagger$, makes sure that the crucial conditions (A4) and (A5) are satisfied.

In the diagrammatic language, where ψ, $\psi^\dagger$, $\partial/\partial\psi$ and $\partial/\partial\psi^\dagger$ are represented by  and  , respectively, we have



The diagrammatic technique makes it possible to by-pass the very tedious algebra involved in the method of comparing coefficients. The required equations are obtained by the much simpler task of collecting all diagrams with the same set of free external lines and setting their algebraic sum to zero. They have been written down in § 3 of II.

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